Optimal Education Policies under Endogenous Borrowing Constraints

Min Wang*

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Abstract

When young students face exogenous borrowing constraints (incomplete markets) on education loans, the simultaneous establishment of a education subsidy and an old-age pension has been shown to restore the complete market allocation (Boldrin and Montes, 2005). If the borrowing constraint is endogenous, owing to limited commitment of repayment and inalienability of future human capital (as in Kehoe and Levine, 1993), Andolfatto and Gervais (2006) by means of an example, argue that, in a small open economy with interest rate exogenously given, the education-subsidy-cum-pension scheme distorts the credit market, and hence, fails to restore the complete market allocation. This paper shows that, in both open economy and closed economy, the complete market allocation can be achieved even with endogenous borrowing constraints. Moreover that policy scheme can remove the indeterminacy issue (multiple equilibria) generated by endogenous borrowing constraints in the general equilibrium set-up. The results broaden the rationale for a two-armed (education and pension) welfare state to a much wider class of economies.

Keywords: Borrowing constraints; Incentive compatability; Education; Intergenerational transfer.

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*Michigan State University, East Lansing MI 48824, USA. E-mail: wangmin@msu.edu

1 Introduction

In growth theory, human capital is deemed an important engine of economic growth (e.g., Lucas, 1988, and Azariadis and Drazen, 1990). [1] More generally, it is beyond controversy that education has a profound beneficial effect on the overall performance of an economy. Yet in most countries, students, especially those from poor families, are generally short of funds for educational investments. Why? Due to the inalienability of human capital, future labor income cannot be collateralized. As such, credit markets severely restrict any borrowing against future human capital for education purposes. [2] Too little human capital is generated, and this constrains overall economic performance. A challenge for development theory emerges from this discussion. Given imperfect credit markets, can public policy restore human capital investment to socially optimal levels?

At first blush, it may appear that a carefully-chosen public subsidy to education could ensure optimal accumulation of human capital. In a recent, important paper, Boldrin and Montes (2005) show that this is generally not enough. They present a three-period overlapping generation model in which the young need to borrow to finance education, middle-aged agents are net lenders, and no borrowing is possible (incomplete markets). The market outcome in this case is clearly inefficient. Would a policy that taxes the middle-aged and makes education-linked transfers to the young work? Boldrin and Montes (2005) show that, while such a policy may improve matters, it may not replicate the complete market allocation. Moreover, such a policy may never get off the ground because the initial middle-aged would be hurt – they would pay into

[1] The cross-country estimation by Barro and Sala-i-Martin (P524, 2003) shows that a 1.3 year increase in male upper–primary-level schooling raises the growth rate by 0.5 percent.

[2] The impact of borrowing constraints on education may not be so evident in high-income countries, such as USA (Cameron and Heckman, 2001; Cameron and Taber, 2004) but is definitely important for poorer countries. Based on cross-country regression analyses, De Gregorio (1996) and Flug et al. (1998) have shown that borrowing constraints limit the education investment. Jacoby (1994), by using household data from Peru, presents similar evidence that children withdraw from school earlier if their family is borrowing constrained.

the system having never received a subsidy from the current old. They go on to show that if credit markets are missing, i.e., people are not allowed to borrow or lend, the only way to restore the efficiency is "establishing publicly financed education and pay-as-you-go pensions simultaneously, and by linking the two flows of payment via the market interest rate". By their setup, the joint institutional arrangements perfectly substitute the missing credit market and therefore can replicate the complete market allocation of human capital investment. Their study provides a rationale for the "cradle to grave" policies that are widely observed (Andolfatto and Gervais, 2006).

Boldrin and Montes (2005), no doubt, provides a deep insight into the welfare state as it pertains to education and pensions; however, it is fair to say that their treatment of the imperfection in the credit market is somewhat arbitrary. They simply assume non-existence of a credit market and impose a zero borrowing limit on the young. Much of the work in the literature on credit market imperfections has focused on relaxing the zero borrowing limit. Human capital investment subject to exogenous borrowing constraints has been studied in papers such as de Gregorio (1996) and Cartiglia (1997). More recently, motivated by Kehoe and Levine (1993), recent studies, like Andolfatto and Gervais (2006), de la Croix and Michel (2007) and Lochner and Mongo-Naranjo (2011) have begun to introduce endogenous borrowing constraints and examine their role in accumulation of human capital. The framework proposed by Kehoe and Levine (1993) and extended in the lifecycle model by Azariadis and Lambertini (2003) has become the de-facto benchmark for analyzing borrowing constraints. In that setup, the borrowing limit arises because the borrower cannot commit to repaying her loan, and if she defaults, the creditor can seize tangible assets but not her private inalienable endowments, such as human capital and government entitlements. Therefore, the only punishment for (or the opportunity cost of) defaulting is being excluded from the credit market for the rest of one's life, and as a consequence, being unable to smoothen consumption. Under perfect information, lenders would set the borrowing limit at the amount where cost and benefit of default are balanced. Hence, any loan less than this borrowing limit is in the borrower's interest to repay, and there is no default in equilibrium.

Evidently, as a natural extension of Boldrin and Montes (2005), one may ask: in the presence of endogenous borrowing constraints, would the Boldrin-Montes education-pension package replicate the complete market allocation? Andolfatto and Gervais (2006) take up this question and demonstrate the possibility that, in a small open economy where interest rate is exogenously given, intergenerational transfer policies tighten the borrowing constraint and leave less resources for the young to invest in human capital. The intuition is that more education subsidies means more tax on the middle-aged and bigger pensions for the old, both of which reduce the need for consumption smoothing, increasing the incentive to default, and resulting in a further-tightened borrowing limit. Andolfatto and Gervais (2006) conclude that there does not exist optimal intergenerational policies with positive education subsidy that replicate the complete market allocation, for "the government subsidy (in education) does not compensate for the contraction in private lending", and the correct policies should tax the young and the old and subsidize the middle-aged.

It is useful to point out that the sharp conclusion of Andolfatto and Gervais (2006) relies entirely on a numerical example, and that example turns out to be somewhat of a special case as this paper shows. It is shown here that, depending on the model's parameters, the upshots of both Andolfatto and Gervais (2006) and Boldrin and Montes (2005) could be correct. How could this be? Recall from above that as the education subsidy increases, an individual's income profile becomes flatter, and consequently, the borrowing limit, as well as the educational investment of a borrowing-constrained individual falls, until the borrowing limit hits zero. At this level of the subsidy, the private market for education loans is completely choked off. If the subsidy is increased further, the borrowing limit remains at zero, and so the entirety of the educational investment is being funded by the subsidy. In particular, the subsidy can rise to the point at which educational investment by the young exactly equals its level in the complete markets case. Note though, that driving the borrowing limit to zero implies optimal savings of the middle-aged agent are forced to bind at zero precluding any consumption smoothing. It is evident that to replicate the complete market allocation, the "right" subsidy must achieve optimal educational investment and consumption smoothing simultaneously. For this to happen, total resources available for educational investment must remain larger than its level in the complete markets case *before* the borrowing limit hits zero. Below, I derive sufficient conditions for the existence of such an optimal subsidy. If the parametric specification of the model satisfies these conditions, the conclusion of Boldrin and Montes (2005) would extend to small open economies with endogenous borrowing constraints. If they do not hold, it is possible that the negative result in Andolfatto and Gervais (2006) would then apply.

This paper also examines the case of a closed economy, where the interest rate is endogenously determined. As shown by Azariadis and Lambertini (2003) and Croix and Michel (2008), once we take account the general equilibrium consequences of endogenous borrowing constraints, multiple equilibria are inevitable. Specifically the general equilibrium outcome is two non-autarkic steady states: one unconstrained with complete market solutions and one constrained. It will be shown below that there *always* exist optimal intergenerational policies with positive education subsidy to restore the complete market solutions when the economy is in non-autarkic constrained steady state equilibrium. In contrast to the case of small open economy, the result does not depend on the parametric specification as long as multiple steady state equilibria are existent. More important, the unconstrained steady state equilibrium is unaffected by the intergenerational policies and for the constrained equilibrium, the new interest rate under the government policies coincides with the one of the unconstrained equilibrium. Therefore Boldrin and Montes's two-armed (education and pension) welfare state not only prevails in the close economy, but also removes the indeterminacy issue stressed in the literature.

The rest of the paper is organized as the follows. The complete market solutions are provided in section 2. The endogenous borrowing limit is determined in section 3. Section 4 examines the optimal intergenerational policies in both small open economy and closed economy. Section 5 concludes. Proofs of all results are contained in the appendices at the end of the paper.

2 Complete markets economy

The model closely follows those described in Boldrin and Montes (2005) and Andolfatto and Gervais (2006). Consider a economy consisting of an infinite sequence of three-period lived overlapping generations, an initial old generation and an initial middle-aged generation. In each generation, there is a continuum of identical members of measure one. Each agent is born with an endowment profile $(\omega_y, \omega_m, \omega_o)$. She invests in human capital when young, works and receives return on that education investment during middle-age and is retired when old. As in Boldrin and Montes (2005), I assume agents draw utility from consumption at middle-age ($c_{m,t}$) and old age ($c_{o,t+1}$). The utility function of an agent born at period $t-1$ is

$$u\left(c_{m,t}\right) + \beta u\left(c_{o,t+1}\right) \tag{1}$$

where β is the subjective discount factor and $u(\cdot)$ is a strictly increasing, concave function and twice continuously differentiable.

When young, an agent invests $x_{t-1} = \omega_y + b_{t-1}$ in human capital (there is no physical capital), in which b_{t-1} is savings if $b_{t-1} < 0$ and borrowings if $b_{t-1} > 0$. I assume $\omega_y = 0$ implying that young agents will need to borrow to finance their education. Middle-aged agents work and earn consumption goods $f(x_{t-1})$ where $f(x_{t-1})$ is the return on their prior education investment and f is a strictly increasing, concave function with $f(0) = 0$. With complete markets, each agent commits to repaying the loan. Her lifecycle budget constraints are

$$
\begin{aligned}
x_{t-1} &= \omega_y + b_{t-1}, && (2)\\
c_{m,t} + s_t &= \omega_m + f(x_{t-1}) - R_t b_{t-1}, && (3)\\
c_{o,t+1} &= \omega_o + R_{t+1} s_t, \text{ and} && (4)\\
0 &\le b_{t-1} \le b_{t-1}^{\max}. && (5)
\end{aligned}
$$

Here R_t is the interest rate between $t-1$ and t, s_t is the savings of middle-aged agent, and $b_{t-1}^{\max}$ is the upper bound of the loan that young can borrow and is defined by the following equation

$$
\omega_m + f\left(b_{t-1}^{\max} + \omega_y\right) - R_t b_{t-1}^{\max} = 0. \tag{6}
$$

For any borrowing $b_{t-1} > b_{t-1}^{\max}$, $\omega_m + f(x_{t-1}) - R_t b_{t-1} < 0$, i.e., the net income of the middle-aged agent is negative.

The first order conditions for the agent's problem are

$$
\begin{aligned}
\frac{u'\left(c_{m,t}^*\right)}{u'\left(c_{o,t+1}^*\right)} &= \beta R_{t+1} && (7)\\
f'\left(x_{t-1}^*\right) &= R_t && (8)
\end{aligned}
$$

In all follows, the superscript $*$ denotes the complete market solutions. Equation (7) equates marginal rate of substitution of consumption to the discounted interest rate. Equation (8) implies that marginal return from investing in human capital should equal the marginal cost of the loan. By (7), we can obtain the explicit solutions of x_{t-1}^* and

$b_{t-1}^{*}:$

$$x_{t-1}^{*} = f'^{-1}(R_t), \tag{9}$$

$$b_{t-1}^{*} = f'^{-1}(R_t) - \omega_y. \tag{10}$$

Finally, in a closed economy, the interest rate is determined by the following general equilibrium condition

$$s_t^{*}\left(R_t^{*}, R_{t+1}^{*}\right) = b_t^{*}\left(R_{t+1}^{*}\right) \tag{11}$$

which, for given initial interest rates (R_{-1}, R_0), clears the credit market for the initial debt. The first-order difference equation (11) characterizes the dynamics of the economy.

3 Borrowing-constrained economy

In this section, I study an economy in which agents cannot commit to repaying their loans and their ability to borrow against future income is limited due by the absence of commitment.

As in Kehoe and Levine (1993), all information is public, and in the event of default, the affected creditors cannot seize the individual's private endowments or education returns, but can appropriate her current and future assets. The only punishment creditors can impose is to keep the defaulter out of credit market for the rest of her life. For borrowers, the cost of default is the foregone lifetime gains from trading in the credit market. Since all information, including the default, is public, creditors allow agents to borrow up to a limit which is in her interests to repay, i.e. for all loan amounts less than that limit, the benefit from trading in the credit market is bigger than the cost of autarkic consumption. As such, default never occurs in equilibrium.

3.1 The Basics

Since all agents borrow when young, it is the choice of the middle-aged agent to default or not. If she contemplates repaying the loan, she faces an optimization problem identical to that in the complete markets economy. Otherwise, she will be excluded from credit market and consume

$$c^d_{m,t} = \omega_m + f\left(b_{t-1} + \omega_y\right) \tag{12}$$

$$c^d_{o,t+1} = \omega_o \tag{13}$$

where the superscript d denotes the allocation in the case of default.

For creditors, the optimal lending decision for agents born at $t-1$ is the solution to the problem that maximizes (1) subject to the budget constraints (2)−(5) and the following individual rationality constraints (IRC)

$$s_t \geq 0 \qquad \text{IRC (1)}$$

$$V_{m,t}\left(\omega, b_{t-1}\right) \geq u\left[\omega_m + f\left(b_{t-1} + \omega_y\right)\right] + \beta u\left(\omega_o\right), \qquad \text{IRC (2)}$$

where

$$V_{m,t}\left(\omega, b_{t-1}\right) \equiv \max_{s_t} \left\{u\left[\omega_m + f\left(b_{t-1} + \omega_y\right) - R_t b_{t-1} - s_t\right] + \beta u\left(s_t R_{t+1} + \omega_o\right)\right\}$$

is the value function of the middle-aged agent who repays the loan and can access the credit market. IRC1 implies she cannot borrow at middle age, for participation in the credit market has no value to her during old age and hence, she will never repay the debt. IRC2 implies that creditors should always offer a loan that makes the borrower prefer repayment to default.

Now, consider the optimization problem of an agent in the borrowing-constrained economy. She takes the borrowing limit $\bar{b}_{t-1}$ as exogenous. If $\bar{b}_{t-1} < b^*_{t-1}$, she is borrowing constrained. Otherwise, her debt constraint is slack. If the latter is the case, the optimality conditions are the same as those in the case of the complete market economy. Otherwise, her first order conditions are

$$b_{t-1}^c = \bar{b}_{t-1} \tag{14}$$

$$f'\left(x_{t-1}^c\right) = R_t + \frac{\lambda_t}{u'\left(c_{m,t}^c\right)} \geq R_t \tag{15}$$

$$\frac{u'\left(c_{m,t}^c\right)}{u'\left(c_{o,t+1}^c\right)} \geq \beta R_{t+1}, \ = \text{ if } s_t^c > 0 \tag{16}$$

where the superscript c denotes the optimal solution of individual in the constrained market, and $\lambda_t > 0$ is the Lagrangian multiplier of the borrowing constraint $b_{t-1} \leq \bar{b}_{t-1}$. Equation (14) implies that, $x_{t-1}^c < x_{t-1}^*$, human capital is under-invested in the imperfect credit market. Equation (15) states that the marginal return from human capital investment is higher than the interest rate. Due to the borrowing constraints, the gains from the investment opportunity cannot be exhausted.

3.2 Endogenous Borrowing Limits

In the following, I examine the individual rationality constraints, IRC1-IRC2, to determine the aforediscussed borrowing limit. First, I characterize the conditions for IRC1, the non-negativity constraint on savings.

Proposition 1 *An agent born at $t-1$ is borrowing-constrained at both period $t-1$ and period t with $\bar{b}_{t-1} = 0$ if and only if $u'\left[\omega_m + f\left(\omega_y\right)\right]/u'\left(\omega_o\right) > \beta R_{t+1}$.*

If condition $u'\left[\omega_m + f\left(\omega_y\right)\right]/u'\left(\omega_o\right) > \beta R_{t+1}$ holds, the middle-aged agent has no incentive to save even if she does not incur any debt when young. Clearly, in this case, the borrowing limit is zero. On the flip side, when $u'\left[\omega_m + f\left(\omega_y\right)\right]/u'\left(\omega_o\right) < \beta R_{t+1}$, the optimal saving of a middle-aged agent without any prior borrowing is positive, i.e.

$s_t^c > 0$. In this case, the borrowing limit is positive, i.e. $\overline{b}_{t-1} > 0$; henceforth I assume, $u'[\omega_m + f(\omega_y)]/u'(\omega_o) < \beta R_{t+1}$ holds which, in turn, guarantees IRC1.[3]

Given this assumption, I proceed to characterize the conditions for the non-default constraint IRC2 and determine the borrowing limit.

Proposition 2 *Denote*

$$H \equiv V_{m,t}(\omega, b_{t-1}) - u[\omega_m + f(b_{t-1} + \omega_y)] - \beta u(\omega_o) \tag{17}$$

(1) the borrowing limit $\overline{b}_{t-1}(R_t, R_{t+1})$ satisfies $H\left(\overline{b}_{t-1}\right) = 0$. If there exist multiple solutions, $\overline{b}_{t-1}$ is equal to the smallest one.

(2) at $\overline{b}_{t-1}$,

$$\frac{\partial H}{\partial b_{t-1}}\Big|_{b_{t-1}=\overline{b}_{t-1}} < 0. \tag{18}$$

Proposition 2 argues that the borrowing limit, $\overline{b}_{t-1}$, should be determined at the point where the benefit of debt default equals its cost; from (18), it is clear that a marginal increase in the borrowing limit would break that balance, and thus violate IRC2.

In equilibrium, creditors would only allow young agents to borrow up to $\overline{b}_{t-1}$, and in their self interest, borrowers would repay the loan in the next period. Unlike the traditional credit rationing models based on asymmetric information between borrowers and lenders, this framework allows the existence of credit rationing even when all information is public; moreover it removes default in equilibrium.

I collect some useful properties of borrowing limit $\overline{b}_{t-1}(R_t, R_{t+1})$,

[3]Note that the condition $u'[\omega_m + f(\omega_y)]/u'(\omega_o) < \beta R_{t+1}$ can be satisfied by parameter specifications that favor savings, such as sufficiently-low ratio of ω_o to ω_m, high β or high intertemporal elasticity of substitution. Intuitively if agent's savings incentive is sufficiently strong at middle age, her cost of defaulting on youthful debt — being excluded from the credit market — would be high and realizing that, creditors would lend to her. Moreover, as will be shown below, the borrowing limit as well as agent's defaulting cost is increasing in the incentive for savings at middle age.

Corollary 1 *Given a sequence of interest rated* $\{R_t\}$*,*

(1) the borrowing limit of youth $\overline{b}_{t-1}$ *is increasing in* R_{t+1} *and decreasing in* R_t*.*

(2) the borrowing limit of youth $\overline{b}_{t-1}$ *is increasing in* ω_y *and* ω_m*, and decreasing in* ω_o*.*

The relationship between borrowing limit and the interest rate, discussed in part (1) of the corollary above, is the same as that in Azariadis and Lambertini (2003) and Croix and Michel (2007).[4] Part (2) of the corollary is crucial to the subsequent analysis because, as will be evident shortly, the intergenerational transfer policies proposed by Boldrin and Montes (2005) may be equivalently expressed as changes in the individual's endowment profile ω_i, $i = \{y, m, o\}$. Since the incentive for middle-aged agents to participate in the credit market is to smooth post-youth consumption, as post-youth endowments flatten, i.e., ω_m decreases or ω_o increases, agents gain less from trade raising their incentive to default, which results in tightened borrowing limits. Of course, increasing ω_y has the same effect as increasing ω_m.[5]

In sum, an agent born at $t-1$ is (not) borrowing-constrained if and only if $\overline{b}_{t-1}(R_t, R_{t+1}) < (\geq) b^*_{t-1}(R_t)$; optimal borrowing of the young is given by $b^c_{t-1} = \min\left\{\overline{b}_{t-1}(R_t, R_{t+1}), b^*_{t-1}(R_t)\right\}$. The interest rate in the borrowing-constrained economy is determined from the following market clearing condition, i.e. given initial interest rates (R_{-1}, R_0)

$$\max\left\{s^c_t\left(R^c_t, R^c_{t+1}\right) - b^c_t\left(R^c_{t+1}, R^c_{t+2}\right),\ s^*_t\left(R^*_t, R^*_{t+1}\right) - b^*_t\left(R^*_{t+1}\right)\right\} = 0 \qquad (19)$$

[4]Intuitively, if current interest rate R_t is high, the debt size carried from youth to middle age is large. As the consequence, the income profile becomes flatter and middle-aged agent has less incentive to smooth consumption and reimburse her loan, leading to tightened borrowing limit. On the contrary, when expected future interest rate R_{t+1} is high, return from participating in credit market becomes high and therefore repaying loans in middle age becomes more attractive.

[5]It is the force of the savings incentive that determines the borrowing limit. It is straightforward to check that the borrowing limit could be raised if discount factor β is higher, human capital more productive or the intertemporal elasticity of substitution is higher. In all these cases, due to the high valuation of consumption smoothing, agents have a strong incentive to repay their loans.

where the savings of middle aged agent are given by $s_t(R_t, R_{t+1}) = s_t^c\left(R_t, R_{t+1}, \overline{b}_{t-1}(R_t, R_{t+1})\right)$ if borrowing constraints always bind on the young, and $s_t(R_t, R_{t+1}) = s_t^*\left(R_t, R_{t+1}, b_{t-1}^*(R_t)\right)$ if borrowing constraints are always slack. Given the stated goals of the paper, I assume $\overline{b}_{t-1}(R_t, R_{t+1}) < b_{t-1}^*(R_t)$ in the following section.

4 Policies

To facilitate the comparison with Andolfatto and Gervais (2006), I will examine optimal policies *at* the (borrowing-constrained) steady state. Such a simplified setting helps obtain sharp results. I begin the discussion in a small open economy facing an (exogenous) interest rate R as in Andolfatto and Gervais (2006). Then I extend the study to the closed-economy case (endogenous interest rate).

4.1 Small open economy

Consider a lump-sum transfer scheme (τ_y, τ_m, τ_o).[6] As discussed in Andolfatto and Gervais (2006), a policy that tries to replicate the complete market solution must satisfy the government budget constraint,

$$\tau_y + \tau_m + \tau_o = 0 \tag{20}$$

and keep the present value lifecycle budget constraint of the agent unchanged

$$\tau_y + \frac{\tau_m}{R} + \frac{\tau_o}{R^2} = 0. \tag{21}$$

[6]In order to compare the results from Andolfatto and Gervais (2006), the government budget would be balanced on a period-by-period basis.

Therefore, the only possible choice of policy scheme that can restore the complete market solution is

$$\tau_m = -(1+R)\,\tau_y \tag{22}$$

$$\tau_o = R\tau_y \tag{23}$$

which is the same as the optimal policy in Boldrin and Montes (2005). Policy 3-tuples (τ_y, τ_m, τ_o) is collapsed to a one-tuple policy choice, τ_y. As will be shown below, any policy that replicates the complete market allocation, must additionally achieve optimal education investment and optimal consumption smoothing. Generically, one policy tool cannot achieve two goals, which may explain why, for some parameterization, optimal policy with $\tau_y > 0$ could be non-existent (as demonstrated in Andolfatto and Gervais, 2006).

Since a lump-sum transfer is equivalent to a re-arrangement of endowment profiles, all analyses reported in the previous section can be applied to this section with ω_i being replaced by $\omega_i' = \omega_i + \tau_i$, for $i = \{y, m, o\}$. The intergenerational transfer policy encourages the middle-aged agent to lend more generously to the young with a commitment of pay back in the form of a pension when old. When the borrowing constraint is exogenous, those transfers can perfectly substitute the missing credit market as in Boldrin and Montes (2003). When the borrowing constraint is endogenous, the government in trying to use intergenerational transfers to substitute the missing credit market distorts margins in the credit market, in particular, the level of the borrowing limit.

Using Corollary 1 and equations (41) and (42), the effect of government policy on the borrowing limit can be expressed as

$$\frac{\partial \overline{b}}{\partial \tau_y} = \frac{\partial \overline{b}}{\partial \omega'_y} - (1+R)\frac{\partial \overline{b}}{\partial \omega'_m} + R\frac{\partial \overline{b}}{\partial \omega'_o}$$
$$= \left[f'\left(\overline{b}+\omega_y+\tau_y\right) - 1 - R\right]\frac{\partial \overline{b}}{\partial \omega'_m} + R\frac{\partial \overline{b}}{\partial \omega'_o}.$$

Setting $\omega_y = 0$ and substituting (42) and (43), we can obtain

$$\frac{\partial \overline{b}}{\partial \tau_y} = \frac{\left[f'\left(\overline{b}+\tau_y\right) - 1 - R\right]\left[u'\left(c_m^c\right) - u'\left(c_m^d\right)\right] + \beta R\left[u'\left(c_o^c\right) - u'\left(c_o^d\right)\right]}{-\left[f'\left(\overline{b}+\tau_y\right) - R\right]u'\left(c_m^c\right) + f'\left(\overline{b}+\tau_y\right)u'\left(c_m^d\right)} \tag{24}$$

Equation (24) is the main analytical result of this paper and from it, I can derive important implications missed in the simulation example of Andolfatto and Gervais (2006).

A most important finding from equation (24) is that, even with endogenous borrowing constraints, the main argument of Boldrin and Montes (2005) could remain valid. Andolfatto and Gervais (2006) make a crowding-out type argument to argue no $\tau_y \geq 0$ implements the complete market allocation: "a one dollar education subsidy may well lead to a reduction in private credit by more than one dollar, leaving the young with less resources than prior to the intervention". Mathematically, that says

$$\frac{\partial x}{\partial \tau_y} = \frac{\partial \overline{b}}{\partial \tau_y} + 1 \leq 0 \text{ or } \frac{\partial \overline{b}}{\partial \tau_y} \leq -1.$$

They show that, as τ_y increases from zero, the income profile becomes flatter, and consequently, the borrowing limit, as well as human capital investment, is monotonically decreasing until $\overline{b} = 0$. At this level of τ_y, the private market for education loans is completely choked off. If τ_y is increased further, the borrowing limit remains at zero, and so the entirety of the human capital investment is funded by the subsidy. In this manner, τ_y can increase until $\tau_y = x^*$.[7] Note though, that driving $\bar{b}$ to 0 implies optimal savings of the middle-aged agent are forced to bind at zero precluding any consumption smoothing.[8] It is evident that to replicate the complete market allocation, the "right" $\tau_y^* \geq 0$ needs to succeed in both dimensions: achieve optimal education investment and consumption smoothing simultaneously. Alternatively, if τ_y increases from zero, total resources available for human capital investment, $\bar{b}(\tau_y) + \tau_y$, must remain larger than x^* *before* the borrowing limit decreases to zero. Below, I will derive sufficient conditions for the existence and non-existence of optimal $\tau_y^* \geq 0$.

Define $\widehat{\tau}_y$ by

$$\frac{u'(\omega_m')}{u'(\omega_o')} = \frac{u'[\omega_m - (1+R)\widehat{\tau}_y + f(\widehat{\tau}_y)]}{u'(\omega_o + R\widehat{\tau}_y)} = \beta R \tag{25}$$

Note that $\widehat{\tau}_y$ adjusts the income endowment to a level that agent carrying no youthful debt prefers autarky at middle age. Then we can apply Proposition 1 and conclude that for any $\tau_y \geq \widehat{\tau}_y$, left hand side of (25) would be greater than βR, leading to zero borrowing limit and binding savings. Therefore the consumption smoothing condition reads $\tau_y^* \leq \widehat{\tau}_y$, and the existence of an optimal subsidy $\tau_y^* \geq 0$ need to satisfy $\bar{b}\left(\tau_y^*\right) + \tau_y^* \geq x^*$ and $\tau_y^* \leq \widehat{\tau}_y$ simultaneously. These two conditions ensures the lump-sum transfers provide enough education funding for youth and substitute the missed credit market without squeezing it.

We note that, in the case of non-existence of optimal $\tau_y^* \geq 0$, the increase of education subsidy cannot compensate the drop of borrowing limit, i.e. $\partial\bar{b}/\partial\tau_y \leq -1$, and therefore resource condition can only be satisfied when consumption smoothing condition is violated, i.e. $\tau_y > \widehat{\tau}_y$. However, $\partial\bar{b}/\partial\tau_y \leq -1$ is not always the case. In the extreme, it can be shown that $\partial\bar{b}/\partial\tau_y > -1$ for all $\tau_y \geq 0$. By equation (24), it is easy

[7] It is easy to check that $\frac{\partial\bar{b}}{\partial\tau_y}\big|_{\bar{b}=0} = 0$. Hence borrowing limit has no response to government policy when it is already zero.

[8] Recall that $\bar{b}$ is determined by $H = 0$ in Proposition 2. If $\bar{b} = 0$, obviously $s_t = 0$.

to check $\partial\left(\partial\overline{b}/\partial\tau_y\right)/\partial\left[f'\left(\overline{b}+\tau_y\right)\right]>0$. Since in the borrowing-constrained economy $f'\left(\overline{b}+\tau_y\right)\geq R$, a low bound of $\partial\overline{b}/\partial\tau_y$ can be derived by substituting $f'\left(\overline{b}+\tau_y\right)=R$ in (24)

$$\frac{\partial\overline{b}}{\partial\tau_y}\geq\frac{1}{R}-\beta\frac{u'\left(\omega_o+R\tau_y\right)}{u'\left[\omega_m-(1+R)\tau_y+f\left(\overline{b}+\tau_y\right)\right]}>\frac{1}{R}-\beta\frac{u'\left(\omega_o\right)}{u'\left[\omega_m+f\left(x^*\right)\right]} \tag{26}$$

from which sufficient conditions on ω or β can be derived to ensure $\partial\overline{b}/\partial\tau_y>-1$ for all $\tau_y\geq 0$. As long as $\partial\overline{b}/\partial\tau_y\leq -1$ does not always hold, existence of optimal $\tau_y^*\geq 0$ is possible.

I go on to derive sufficient condition on ω_m to guarantee existence of optimal $\tau_y^*\geq 0$. Similar argument can apply for other parameters. First we need to characterize the upper bound of ω_m, which is define by $\overline{b}\left(\overline{\omega}_m\right)=x^*$, i.e.

$$\underset{s}{Max}\left\{u\left[\overline{\omega}_m+f\left(x^*\right)-Rx^*-s\right]+\beta u\left(\omega_o+sR\right)\right\}-u\left[\overline{\omega}_m+f\left(x^*\right)\right]-\beta u\left(\omega_o\right)=0 \tag{27}$$

According to Corollary 1 borrowing limit is increasing in ω_m. The above definition implies that for any $\omega_m\geq\overline{\omega}_m$, the borrowing constraint is relaxed by $\overline{b}\left(\omega_m\right)>x^*$. For all meaningful discussion, ω_m should be less than $\overline{\omega}_m$ such that the economy is initially borrowing constrained. Next, I define $\widehat{\omega}_m$ by

$$\frac{u'\left[\widehat{\omega}_m-(1+R)x^*+f\left(x^*\right)\right]}{u'\left(\omega_o+Rx^*\right)}=\beta R. \tag{28}$$

Comparing (25) to (28), we can learn that $\widehat{\omega}_m$ is the endowment level that equates $\widehat{\tau}_y$ to x^*. As presented in the following Proposition, $\widehat{\omega}_m$ is the threshold value for the existence of optimal $\tau_y^*\geq 0$.

Proposition 3 *(1) Suppose $\overline{\omega}_m\geq\widehat{\omega}_m$, optimal $\tau_y^*\geq 0$ replicating complete market* *solutions exists if and only if* $\omega_m \in [\widehat{\omega}_m, \overline{\omega}_m)$.

(2) If $\overline{\omega}_m < \widehat{\omega}_m$*, there does not exist optimal* $\tau_y^* \geq 0$.

Because ω_m are ω_o are symmetric, similar conditions can be derived for ω_o which is required to be low enough. Furthermore, since by (25) $\widehat{\tau}_y$ is increasing in β, we can define $\overline{\beta}$ and $\widehat{\beta}$ in the same way such that optimal $\widehat{\tau}_y \geq 0$ exists if and only if $\beta \in \left[\widehat{\beta}, \min\left\{\overline{\beta}, 1\right\}\right)$ and $\widehat{\beta} < \overline{\beta}$.

Now from Proposition 3 we can conclude that if and only if agents have sufficiently high incentive to smooth consumption, does there exist an optimal $\tau_y^* \geq 0$ capable of replicating the complete market solution. Intuitively, when the consumption-smoothing motive is strong, the agent has less incentive to default, and thus can get a relatively generous borrowing limit from the creditors. As the government increases the education subsidy, the individual's borrowing limit falls. But since the agent's initial borrowing limit is abundant, a fairly large education subsidy is required to drive the borrowing limit down to zero. It is possible that the rate at which the borrowing limit falls is less than growth rate of the education subsidy and, as such, the total resource available to the young — education subsidy plus borrowing limit — could rise before the borrowing limit reaches zero. In this case, the young can afford optimal human capital investment, x^*, and achieve consumption smoothing simultaneously.

In the following, I will use a numerical example to demonstrate the results of Proposition 3. The parametric specification used is as follows: $f(x) = 1.3x^{0.8}$, $u(\cdot) = \ln(\cdot)$, $R = 1.2$, $\beta = 0.66$, and $(\omega_y, \omega_o) = (0, 1)$. Then we can compute $\widehat{\omega}_m = 2.35$ and $\overline{\omega}_m = 3.13$. Hence for any $\omega_m \in [2.35, 3.13)$, there exists optimal $\tau_y^* \geq 0$. Figure 1 and 2 respectively demonstrate the borrowing limit and human capital investment in the economy with $\omega_m = 2$ and $\omega_m = 2.8$. In both Figures, borrowing limit is monotonically decreasing in τ_y.[9] Figure 1 illustrates the case that borrowing limit decrease to zero

[9]Depending on parameter specification, education investment could be decreasing, non-decreasing or decreasing first and then increasing in τ_y.

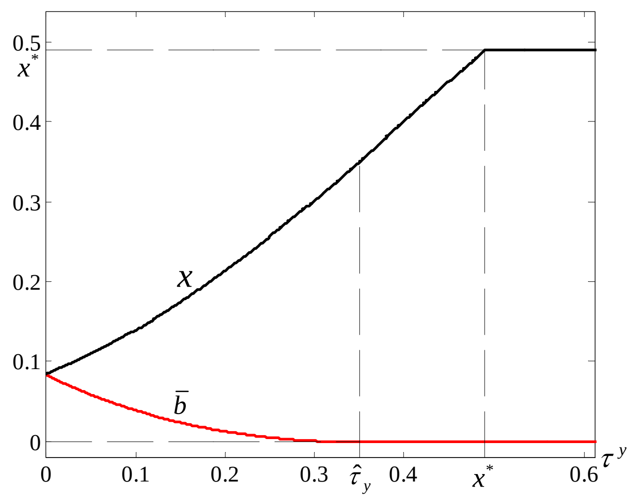

Figure 1: Non-existence of optimal $\tau_y^* > 0$

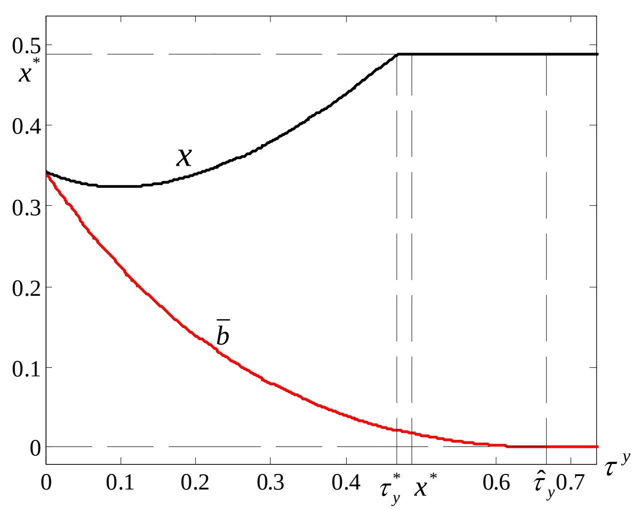

Figure 2: Existence of optimal $\tau_y^* > 0$

before human capital investment achieves the optimal level x^*, and therefore optimal $\tau_y^* \geq 0$ does not exist. In contrast, Figure 2 exhibits that agent can obtain optimum for $\forall \tau_y \in \left[\tau_y^*, \widehat{\tau}_y\right]$.

4.2 Closed economy

This section explores a closed economy where the interest rate is endogenously determined instead of being exogenously given by international credit market as in the open economy. Under such a general equilibrium framework, a most striking result of incorporating endogenous borrowing constraints in finite lifecycle economies is the existence of multiple steady states and thus complex dynamics (Azariadis and Lambertini, 2003; Croix and Michel, 2008). In the following, I show that the optimal scheme designed in the open economy case not only prevails in the close economy, but also could remove the issue of indeterminacy or multiple steady states stressed in the literature.

To development the argument, we firstly need to solve the general equilibrium solutions. As discussed in Section 3.2, the steady state aggregate asset demands are

$$D^*(R) = s^*(R, R) - b^*(R) \tag{29}$$

if borrowing constraints are slack; and

$$D^c(R) = s^c(R, R) - \overline{b}(R, R) \tag{30}$$

if borrowing constraints are binding. As in (19), the steady state equilibrium interest rate in the closed economy is defined by

$$\max\{D^*(R^*),\ D^c(R^c)\} = 0 \tag{31}$$

Following Azariadis and Lambertini (2003), I define the unique boundary interest rate $\widehat{R}$ by

$$\bar{b}\left(\widehat{R},\widehat{R}\right)=b^{*}\left(\widehat{R}\right) \tag{32}$$

such that the stationary aggregate asset demand are borrowing constrained if and only if $R<\widehat{R}$.[10] Moreover, by defining

$$R_{\min}=\frac{u'\left[\omega_{m}+f\left(\omega_{y}\right)\right]}{\beta u'\left(\omega_{o}\right)} \tag{33}$$

and applying Proposition (1), the economy has zero borrowing limit and thus is in an autarkic equilibrium if $R<R_{\min}$. Finally we have a continuous aggregate asset demand function

$$D\left(R\right)=\begin{cases}0, & \text{if } R\leq R_{\min}\\ D^{c}\left(R\right), & \text{if } R\in\left(R_{\min},\widehat{R}\right)\\ D^{u}\left(R\right), & \text{if } R>\widehat{R}\end{cases} \tag{34}$$

Azariadis and Lambertini (2003) have solved the general equilibrium solutions to $D\left(R\right)=0$. Without proof, I summarize their results in the following

Proposition 4 *If $\widehat{R}<R^{*}$ and the consumption at middle age and old age are gross substitutes, there exist three steady state equilibria*

(1) one autarkic borrowing constrained equilibrium with zero asset holdings at any $R<R_{\min}$;

(2) one borrowing constrained non-autarkic equilibrium with non-zero asset holdings at $R^{c}\in\left(R_{\min},\widehat{R}\right)$;

(3) one unconstrained complete market equilibrium at $R^{*}>\widehat{R}$.

[10] See Azariadis and Lambertini (2003) for the proof of unique $\widehat{R}$.

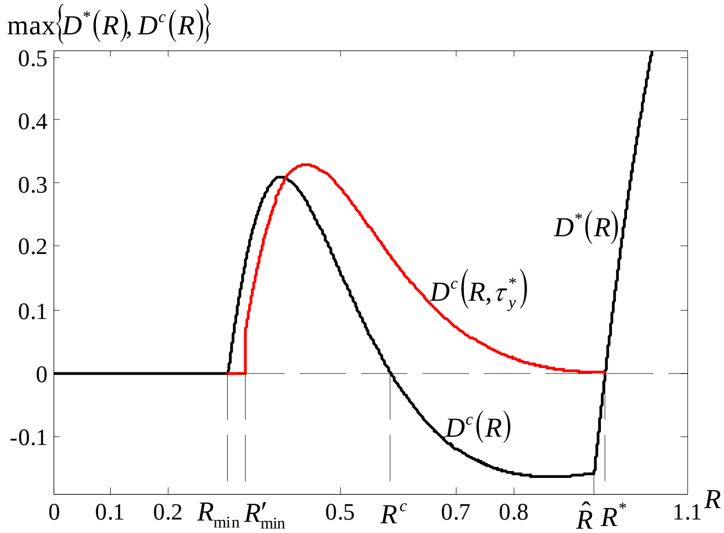

Figure 3: Steady state aggregate asset demand

Continuing with the numerical example, I illustrate the results of Proposition 4 in Figure 3.[11] It is easy to check $dD^c(R)/dR|_{R=R_{\min}} > 0$. Therefore the assumption $\widehat{R} < R^*$ guarantees the existence of borrowing constrained non-autarkic equilibrium R^c.[12] The gross consumption substitution between the old and the middle aged guarantees that the savings in the complete market increase in the interest rate. Since $b^*(R)$ equals $f'^{-1}(R)$ and thus monotonically decreases in R, the gross substitution assumption finally ensures $D^*(R)$ to monotonically increase in R.

We now can turn our attention to the question that how the policy scheme proposed by Boldrin and Montes (2003) can be applied to the general equilibrium case. We note first that the key feature of the afore-discussed policy scheme is to keep the present value lifecycle budget constraint of the agent unchanged. That implies, under the government transfer policies, the complete market solutions of agent's total education investment and total savings remain the same. If the credit market is complete, the government

[11]In this example, all parameters are kept the same except $\omega_m = 5$ and interest rate R is endogenously determined.

[12]If we want the non-autarkic constrained equilibrium R^c to be unique, an additional assumption $dD^c(R)/dR|_{R=R^c} < 0$ is required.

transfers become the perfect substitutes for private education investment or savings, and their only role is to crowd out private education investment and savings by the same amount, i.e. the agent's optimal savings and borrowing are respectively equal to $s^* (R) - \tau_y$ and $b^* (R) - \tau_y$. Such a policy effect would lead to an important result that, in the complete market, the aggregate asset demand $D^* (R)$, the general equilibrium condition and the equilibrium solution all remain the same as before. Looking at Figure 3, that result means curve $D^* (R)$ would keep unchanged for any $\tau_y < b^*$. Therefore we can conclude that the complete market equilibrium R^* can be supported by any $\tau_y < b^*$.

We now need to investigate, under the government intergenerational transfer policy, what would happen for aggregate asset demand in constrained case $D^c (R)$. It is difficult to get analytical results of $\partial D^c / \partial \tau_y$. Using the simulation example, however, we can find that the government subsidy τ_y could move the curve $D^c (R)$ to the right and if τ_y is large enough, eventually $\widehat{R}$ would coincide with R^*, eliminating the non-autarkic constrained equilibrium. In Figure 3, $D^c \left(R, \tau_y^*\right)$ demonstrates the curve under optimal $\tau_y^* > 0$.

In the following we can show the optimal $\tau_y^* > 0$, illustrated in 3, always exists for the non-autarkic constrained equilibrium. First, according to Proposition 4, the existence of non-autarkic constrained equilibrium means $\widehat{R} < R^*$. Second, by definition of $\widehat{R}$, if $\widehat{R} < R^*$, we have

$$\bar{b} (R^*, R^*) > b^* (R^*) .$$

Finally since by the discussion in the case of open economy, $\partial \bar{b} / \partial \tau_y < 0$ and if τ_y is sufficiently large, $\bar{b} = 0$, there should always exist an optimal $\tau_y^* (R^*) > 0$, defined by

$$\bar{b} \left(R^*, R^*, \tau_y^*\right) = b^* (R^*) , \tag{35}$$

such that borrowing constraints become relaxed for the agent. Accordingly under the optimal τ_y^*, the savings for the agent with borrowing constraints equal the savings in the complete market, i.e. $s^c\left(R^*, R^*, \tau_y^*\right) = s^*\left(R^*, R^*\right)$. Hence the new equilibrium with borrowing constraints coincides with the unconstrained equilibrium and the two non-autarkic equilibria are reduced to one. Evidently in the general equilibrium set-up, the education-subsidy-cum-pension scheme, proposed by Boldrin and Montes (2005), not only can restore the complete market solutions when the economy is in the non-autarkic constrained equilibrium, but also could eliminate the issue of multiple non-autarkic equilibria.

5 Conclusion

Imperfect credit market constrains the investment on human capital. Intergenerational transfers that subsidize the education of young and pension of old simultaneously is verified to be the optimal policies to replicate complete market allocation by Boldrin and Montes (2005). With respect to their study, I, following Kehoe and Levine (1993), introduce the endogenous borrowing limit in a three period OLG model with human capital investment.

The endogenous borrowing limit arises because people can not commit to repay their loan and creditors can not garnish the return of human capital. Comparing to Andolfatto and Gervais (2006) who use a numerical example to show the non-existence of optimal intergenerational policies in a small open economy, I derive the analytical results and demonstrate that results of Boldrin and Montes (2005) could still be valid in the setting of endogenous borrowing limit.

Consider the structure of intergenerational transfers proposed by Boldrin and Montes (2005), the education subsidy expands borrowing limit, but in the meanwhile, tax on middle-aged agent and social pension tighten the constraints. The effect of intergenerational transfers on borrowing limit and human capital investment is not straightforward. I have shown in this paper that, in a small open economy with exogenous interest rate, if individual savings incentive is sufficiently high, there could exist optimal intergenerational transfers to replicate the complete market solutions.

Moreover, the study of the case of closed economy, where interest rate is endogenously determined, shows there always exists optimal intergenerational transfers to replicate the complete market solutions when the economy is in the non-autarkic constrained equilibrium and thus reduce the two non-autarkic equilibrium to one. Therefore, the two-armed (education and pension) welfare state, proposed by Boldrin and Montes (2005), is broadend to a much wider class of economies.

Appendix A

Proof of Proposition 1. The sufficiency part relies on definition of constrained demand of assets. Firstly by definition, agent born at $t-1$ being borrowing-constrained at t means $u'\left(c^c_{m,t}\right)/u'\left(c^c_{o,t+1}\right) > \beta R_{t+1}$ such that optimal savings at middle-age are binding, i.e. $s^c_t = 0$. In addition, by the condition $\overline{b}_{t-1} = 0$, we have $c^c_{m,t} = \omega_m + f\left(\omega_y\right)$ and $c^c_{o,t+1} = \omega_o$. Therefore $u'\left[\omega_m + f\left(\omega_y\right)\right]/u'\left(\omega_o\right) > \beta R_{t+1}$.

To prove the necessity part, we know that for all $b_{t-1} \in \left(0, \min\left\{\overline{b}_{t-1}, b^*_{t-1}\right\}\right]$,

$$\frac{\partial\left[f\left(\omega_y + b_{t-1}\right) - R_t b_{t-1}\right]}{\partial b_{t-1}} = f'\left(\omega_y + b_{t-1}\right) - R_t \geq 0 \tag{36}$$

given any possible borrowing limit $\overline{b}_{t-1} > 0$. Since $u\left(\cdot\right)$ is concave and twice continuous, the condition $u'\left[\omega_m + f\left(\omega_y\right)\right]/u'\left(\omega_o\right) > \beta R_{t+1}$ ensures that there must exist some $\widetilde{b}_{t-1} \in \left(0, \min\left\{\overline{b}_{t-1}, b^*_{t-1}\right\}\right]$ such that

$$\frac{u'\left[\omega_m + f\left(\omega_y\right)\right]}{u'\left(\omega_o\right)} \geq \frac{u'\left[\omega_m + f\left(\omega_y + \widetilde{b}_{t-1}\right) - R_t\widetilde{b}_{t-1}\right]}{u'\left(\omega_o\right)} > \beta R_{t+1}$$

Therefore for all $b_{t-1} \leq \widetilde{b}_{t-1}$, the optimal savings s_t are binding and equal to zero, which means that middle-aged agent prefers autarky and would default the youthful debt. Given that information, creditor would not set any strictly positive borrowing limit. Hence $\overline{b}_{t-1} = 0$ and, by $u'\left[\omega_m + f\left(\omega_y\right)\right]/u'\left(\omega_o\right) > \beta R_{t+1}$, $s^c_t = 0$. ■

Proof of Proposition 2. To prove part 1, we need to prove existence of solution $\overline{b}_{t-1}\left(R_t, R_{t+1}\right)$ and for all $b_{t-1} \in \left[0, \overline{b}_{t-1}\right]$, both IRC1 and IRC2 hold. First we note H is continuously differentiable with $b_{t-1} \in \left[0, b^{\max}_{t-1}\right]$ and it is easy to check that $H\big|_{b_{t-1}=0} > 0$ and $H\Big|_{b_{t-1}=b^{\max}_{t-1}} < 0$. Therefore, H intersects the line $b_{t-1} = 0$ from above at least once and, if there are more than one intersection, as illustrated in Figure 4, H intersects b_{t-1} axis from above and below alternatively with the first and last

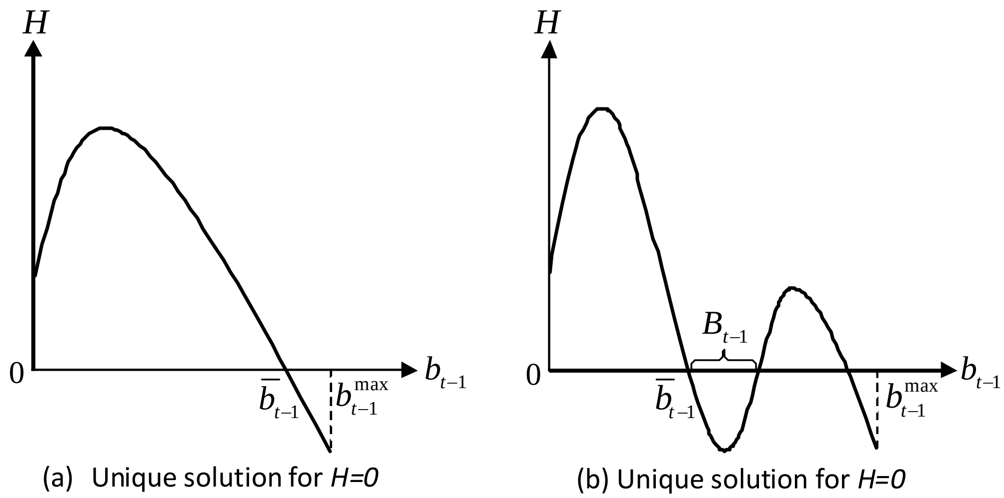

Figure 4: Possible solutions of $\overline{b}_{t-1}$

one intersected from above. Therefore, there exists either a unique solution $\overline{b}_{t-1}$ as Figure 4.(a) shows such that for all $b_{t-1} \in \left[0, \overline{b}_{t-1}\right]$, $H \geq 0$, i.e. IRC2 holds, or multiple solutions for $H = 0$ as in Figure 4.(b). For the case of multiple solutions, $\overline{b}_{t-1}$ is defined by the smallest solution. Otherwise, there always exists a subset $B_{t-1} \subset \left[0, \overline{b}_{t-1}\right]$ such that for any $b_{t-1} \in B_{t-1}$, $H < 0$ and IRC2 is violated.

It remains to verify IRC1, i.e. $s_t^c \geq 0$ for all $b_{t-1} \in \left[0, \overline{b}_{t-1}\right]$, which is equivalent to

$$\frac{u'\left[\omega_m + f\left(\omega_y + b_{t-1}\right) - R_t b_{t-1}\right]}{u'\left(\omega_o\right)} < \beta R_{t+1}$$

Since $u'\left[\omega_m + f\left(\omega_y\right)\right] / u'\left(\omega_o\right) < \beta R_{t+1}$ and $u\left(\cdot\right)$ is concave, it is enough to show, for all $b_{t-1} \in \left[0, \overline{b}_{t-1}\right]$,

$$f\left(\omega_y + b_{t-1}\right) - R_t b_{t-1} > f\left(\omega_y\right) \tag{37}$$

Recall that optimal borrowing b_{t-1}^* is defined by $f'\left(\omega_y + b_{t-1}^*\right) = R_t$. Therefore, for all $b_{t-1} \leq b_{t-1}^*$, $f\left(\omega_y + b_{t-1}\right) - R_t b_{t-1}$ is increasing in b_{t-1} and greater than $f\left(\omega_y\right)$. We thus prove (37) for all $b_{t-1} \in \left[0, b_{t-1}^*\right]$ and only need to verify (37) for any $b_{t-1} \in \left[b_{t-1}^*, \overline{b}_{t-1}\right]$

if $\bar{b}_{t-1} > b^*_{t-1}$. Since $f(\omega_y + b_{t-1}) - R_t b_{t-1}$ is monotonically decreasing in b_{t-1} for $b_{t-1} \in \left[b^*_{t-1}, \bar{b}_{t-1}\right]$ and $u(\cdot)$ is concave, it is enough to show

$$\frac{u'\left[\omega_m + f\left(\omega_y + \bar{b}_{t-1}\right) - R_t \bar{b}_{t-1}\right]}{u'(\omega_o)} < \beta R_{t+1}$$

In the following, we can show it by contradiction.

Suppose $u'\left[\omega_m + f\left(\omega_y + \bar{b}_{t-1}\right) - R_t \bar{b}_{t-1}\right] / u'(\omega_o) \geq \beta R_{t+1}$. Then given youthful debt $\bar{b}_{t-1}$, optimal savings of middle-aged agent is $\widetilde{s}^*_t \leq 0$ and we can define $V_{m,t}$ as

$$V_{m,t} = u\left[\omega_m + f\left(\omega_y + \bar{b}_{t-1}\right) - R_t \bar{b}_{t-1} - \widetilde{s}^*_t\right] + \beta u\left(\widetilde{s}^*_t R_{t+1} + \omega_o\right)$$

Moreover, by using $u'[\omega_m + f(\omega_y)]/u'(\omega_o) < \beta R_{t+1}$, we can obtain $f\left(\omega_y + \bar{b}_{t-1}\right) - R_t \bar{b}_{t-1} < f(\omega_y)$ and thus

$$V_{m,t} < u\left[\omega_m + f(\omega_y) - \widetilde{s}^*_t\right] + \beta u\left(\widetilde{s}^*_t R_{t+1} + \omega_o\right) \tag{38}$$

Since $u'[\omega_m + f(\omega_y)]/u'(\omega_o) < \beta R_{t+1}$, for the middle-aged agent with income $\omega_m + f(\omega_y)$ and without debt, her optimal savings should be strictly positive. In addition, by second order sufficient condition that $u[\omega_m + f(\omega_y) - s_t] + \beta u(s_t R_{t+1} + \omega_o)$ is concave in s_t, that middle-aged agent's utility value at the point $s_t = 0$ should be greater than or equal to that at the point $\widetilde{s}^*_t \leq 0$

$$u\left[\omega_m + f(\omega_y) - \widetilde{s}^*_t\right] + \beta u\left(\widetilde{s}^*_t R_{t+1} + \omega_o\right) \leq u\left[\omega_m + f(\omega_y)\right] + \beta u(\omega_o) \tag{39}$$

By combining (38) and (39), finally we have

$$V_{m,t} < u\left[\omega_m + f(\omega_y)\right] + \beta u(\omega_o) < u\left[\omega_m + f\left(\omega_y + \bar{b}_{t-1}\right)\right] + \beta u(\omega_o) \tag{40}$$

However, according to the definition of $\overline{b}_{t-1}$, $V_{m,t} = u\left[\omega_m + f\left(\omega_y + \overline{b}_{t-1}\right)\right] + \beta u\left(\omega_o\right)$ which contradicts with (40).

Part 2 follows directly from Figure 4. ■

Proof of Corollary 1.

Applying implicit function theorem and envelop theorem in equation (17), it is straightforward to have

$$\begin{aligned}
\frac{\partial \overline{b}_{t-1}}{\partial R_t} &= -\frac{-\overline{b}_{t-1} u'\left(c^c_{m,t}\right)}{\left.\frac{\partial H}{\partial b_{t-1}}\right|_{b_{t-1}=\overline{b}_{t-1}}} < 0 \\
\frac{\partial \overline{b}_{t-1}}{\partial R_{t+1}} &= -\frac{s^c_t u'\left(c^c_{o,t}\right)}{\left.\frac{\partial H}{\partial b_{t-1}}\right|_{b_{t-1}=\overline{b}_{t-1}}} > 0
\end{aligned}$$

Similarly, we can obtain

$$\frac{\partial \overline{b}_{t-1}}{\partial \omega_y} = -\frac{f'\left(\overline{b}_{t-1} + \omega_y\right)\left[u'\left(c^c_{m,t}\right) - u'\left(c^d_{m,t}\right)\right]}{\left.\frac{\partial H}{\partial b_{t-1}}\right|_{b_{t-1}=\overline{b}_{t-1}}} > 0 \tag{41}$$

$$\frac{\partial \overline{b}_{t-1}}{\partial \omega_m} = -\frac{u'\left(c^c_{m,t}\right) - u'\left(c^d_{m,t}\right)}{\left.\frac{\partial H}{\partial b_{t-1}}\right|_{b_{t-1}=\overline{b}_{t-1}}} > 0 \tag{42}$$

$$\frac{\partial \overline{b}_{t-1}}{\partial \omega_o} = -\frac{\beta u'\left(c^c_{o,t}\right) - \beta u'\left(c^d_{o,t}\right)}{\left.\frac{\partial H}{\partial b_{t-1}}\right|_{b_{t-1}=\overline{b}_{t-1}}} < 0 \tag{43}$$

where $c^c_{m,t} = \omega_m + f\left(b_{t-1} + \omega_y\right) - R_t b_{t-1} - s^c_t$, $c^d_{m,t} = \omega_m + f\left(b_{t-1} + \omega_y\right)$, $c^c_{o,t} = s^c_t R_{t+1} + \omega_o$ and $c^d_{o,t} = \omega_o$. Since $c^c_{m,t} < c^d_{m,t}$ and $s^c_t \geq 0$, the signs of $\partial \overline{b}_{t-1}/\partial \omega_i$, $i = \{y, m, o\}$, are straightforward. ■

Proof of Proposition 3. It is evident that Part 2 follows directly from Part 1. Therefore we only need to prove Part 1. Define total resource available for young agent x by a function $F\left(\tau_y\right)$, i.e. $x = F\left(\tau_y\right) = \overline{b}\left(\tau_y\right) + \tau_y$. Then to prove necessity part, we need to prove that for all $\omega_m \in \left[\widehat{\omega}_m, \overline{\omega}_m\right)$, there exists τ_y such that both resource condition, i.e. $F\left(\tau_y\right) \geq x^*$, and consumption smoothing condition, i.e. $\tau_y \leq \widehat{\tau}_y$, are

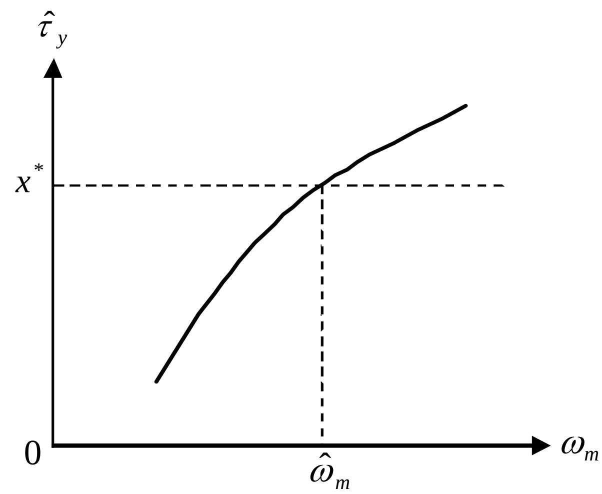

Figure 5: The relationship between $\widehat{\tau}_y$ and ω_m.

satisfied. Note, by definition $F(\tau_y) \geq \tau_y$ always holds. Therefore if we can prove $\widehat{\tau}_y(\omega_m) > x^*$ for all $\omega_m \in [\widehat{\omega}_m, \overline{\omega}_m)$, we would have $F(\tau_y)\big|_{\tau_y=\widehat{\tau}_y} = \widehat{\tau}_y \geq x^*$ for all $\omega_m \in [\widehat{\omega}_m, \overline{\omega}_m)$, and then by using $F(\tau_y)\big|_{\tau_y=\widehat{\tau}_y} \geq x^* > F(\tau_y)\big|_{\tau_y=0}$ and intermediate value theorem, we can conclude that for any $\omega_m \in [\widehat{\omega}_m, \overline{\omega}_m)$ there must exist $\tau_y \in (0, \widehat{\tau}_y]$ such that $F(\tau_y) = x^*$, which proves the necessity part.

We then prove $\widehat{\tau}_y(\omega_m) > x^*$ for all $\omega_m \in [\widehat{\omega}_m, \overline{\omega}_m)$. Firstly by applying implicit function theorem in (25), we can obtain

$$\frac{\partial \widehat{\tau}_y}{\partial \omega_m} = -\frac{u''(\omega_m')\,u'(\omega_o')}{u''(\omega_m')\,u'(\omega_o')\,[f'(\widehat{\tau}_y) - 1 - R] - Ru''(\omega_o')}$$

Evidently if $f'(\widehat{\tau}_y) \leq (1+R)$, $\partial\widehat{\tau}_y/\partial\omega_m > 0$. Secondly, according to (25) and (28), we know $\widehat{\tau}_y\,|_{\omega_m=\widehat{\omega}_m} = x^*$. Therefore $f'(\widehat{\tau}_y)\,|_{\omega_m=\widehat{\omega}_m} = (1+R)$ and we have

$$\frac{\partial \widehat{\tau}_y}{\partial \omega_m}\,|_{\omega_m=\widehat{\omega}_m} > 0$$

Third, by (25) and (28), the solution of ω_m to $\widehat{\tau}_y(\omega_m) = x^*$ is unique and equal to $\widehat{\omega}_m$. I illustrate these afore-discussed relationship between $\widehat{\tau}_y$ and ω_m in Fig. 5. Evidently

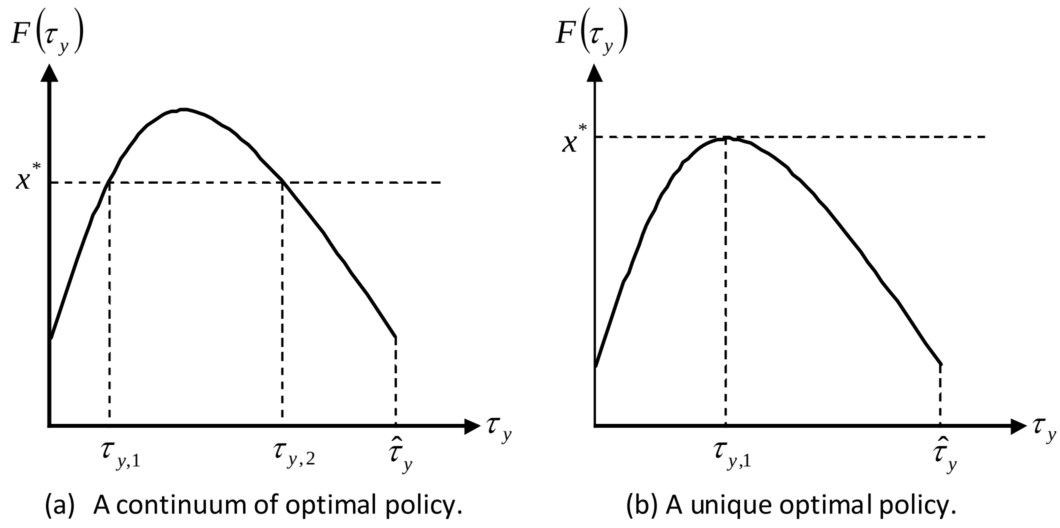

Figure 6: Two cases for the subset $[\tau_{y,1}, \tau_{y,2}]$

if and only if $\omega_m \in (\widehat{\omega}_m, \overline{\omega}_m)$, $\widehat{\tau}_y(\omega_m) \geq x^*$. Otherwise, $\widehat{\tau}_y$ would intersect with the line $\widehat{\tau}_y = x^*$ more than once in Fig. 5 and $\widehat{\tau}_y(\omega_m) = x^*$ has multiple solutions of ω_m, which is impossible as discussed.

The sufficiency part is equivalent to the statement that optimal $\tau_y^* \geq 0$ does not exist if $\omega_m \in (\omega_o, \widehat{\omega}_m)$, where ω_o is by assumption the low bound of ω_m. I prove it by contradiction. Firstly note $\widehat{\tau}_y(\omega_m) < x^*$ for any $\omega_m \in (\omega_o, \widehat{\omega}_m)$. Therefore we have $F(\tau_y)\left|_{\tau_y=\widehat{\tau}_y}\right. = \widehat{\tau}_y < x^*$ for any $\omega_m \in (\omega_o, \widehat{\omega}_m)$. Now suppose optimal $\tau_y^* \geq 0$ exists for some $\widetilde{\omega}^m \in (\omega_o, \widehat{\omega}_m)$. Recall existence of optimal $\tau_y^* \geq 0$ requires resource condition $F(\tau_y^*) \geq x^*$ and consumption smoothing condition $\tau_y^* < \widehat{\tau}_y$. Since $F(\tau_y)\left|_{\tau_y=\widehat{\tau}_y}\right. < x^*$ holds for any $\omega_m \in (\omega_o, \widehat{\omega}_m)$, for $\widetilde{\omega}^m$, there must exist a subset $[\tau_{y,1}, \tau_{y,2}] \subset (0, \widehat{\tau}_y)$ such that for all $\tau_y \in [\tau_{y,1}, \tau_{y,2}]$, complete market solution can be restored, i.e. $F(\tau_y) \geq x^*$, and $F(\tau_y)\left|_{\tau_y=\tau_{y,1}}\right. = F(\tau_y)\left|_{\tau_y=\tau_{y,2}}\right. = x^*$. The set $[\tau_{y,1}, \tau_{y,2}]$ has two possible cases, a continuum of optimal policy $\tau_{y,1} < \tau_{y,2}$ and a unique optimal policy $\tau_{y,1} = \tau_{y,2}$ as shown in Fig. 6. Their corresponding proofs are different.

If $\tau_{y,1} < \tau_{y,2}$, then we must have $F'(\tau_y)\left|_{\tau_y=\tau_{y,1}}\right. > 0$ and $F'(\tau_y)\left|_{\tau_y=\tau_{y,2}}\right. < 0$. Moreover, since $\tau_{y,1}$ and $\tau_{y,2}$ replicate complete market solution, for $\tau_y = \{\tau_{y,1}, \tau_{y,2}\}$,

$u'\left(c_{m,t}^{c}\right)=\beta Ru'\left(c_{o,t+1}^{c}\right)$ and $f'\left(\bar{b}+\tau_y\right)=f'\left(x^*\right)=R$. Substituting these two equations in (24), we have, for $\tau_y=\{\tau_{y,1},\tau_{y,2}\}$,

$$F'\left(\tau_y\right)=1+\frac{\partial\bar{b}}{\partial\tau_y}=1+\frac{1}{R}-\beta\frac{u'\left(\omega_o+R\tau_y\right)}{u'\left[\omega_m-(1+R)\tau_y+f\left(x^*\right)\right]}$$

and $F''\left(\tau_y\right)>0$. Therefore, if $F'\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}>0$, then we must have $F'\left(\tau_y\right)\big|_{\tau_y=\tau_{y,2}}>0$ which contradicts with the fact $F'\left(\tau_y\right)\big|_{\tau_y=\tau_{y,2}}<0$. If $\tau_{y,1}=\tau_{y,2}=x^*$, then $F\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}$ must be a local maximum as illustrated in Fig. 6, i.e. $F'\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}=0$ and $F''\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}\leq 0$. By using (7), $F'\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}=0$ and $f'\left(\tau_y+\bar{b}\right)\big|_{\tau_y=\tau_{y,1}}=R$, some tedious algebra yields

$$F''\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}=\frac{Ru'\left(c_m^d\right)\left\{\begin{array}{c}-u''\left(c_m^*\right)\left(R-1-\frac{\partial s^*}{\partial\tau}\right)-(1+R)\,u''\left(c_m^d\right)\\+\beta R\left[u''\left(c_o^*\right)\left(R+R\frac{\partial s^*}{\partial\tau}\right)-u''\left(c_o^d\right)R\right]\end{array}\right\}}{\left[f'\left(x^*\right)u'\left(c_m^d\right)\right]^2}$$

Since $u'\left(\omega_o+R\tau_{y,1}+Rs^*\right)/u'\left[\omega_m+f\left(x^*\right)-(1+R)\tau_{y,1}-R\bar{b}-s^*\right]=\beta R$, we have

$$\frac{\partial s^*}{\partial\tau_y}\big|_{\tau_y=\tau_{y,1}}=-1$$

By substituting the above result into $F''\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}$, we finally get

$$F''\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}=-\frac{Ru'\left(c_m^d\right)\left[u''\left(c_m^*\right)R+(1+R)\,u''\left(c_m^d\right)+\beta R^2u''\left(c_o^d\right)\right]}{\left[f'\left(x^*\right)u'\left(c_m^d\right)\right]^2}>0$$

which contradicts with the fact $F''\left(\tau_y\right)\big|_{\tau_y=\tau_{y,1}}\leq 0$. ■

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